

Infinite Operator-Sum Representation of Density Operator for a Dissipative Cavity with Kerr Medium Derived by Virtue of Entangled State Representation

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Abstract By using the thermo entangled state representation we solve the master equation for a dissipative cavity with Kerr medium to obtain density operators' infinite operator-sum representation $\rho(t) = \sum_{m,n,l=0}^{\infty} M_{m,n,l} \rho_0 M_{m,n,l}^\dagger$. It is noticeable that $M_{m,n,l}$ is not Hermite conjugate to $M_{m,n,l}^\dagger$, nevertheless the normalization $\sum_{m,n,l=0}^{\infty} M_{m,n,l}^\dagger M_{m,n,l} = 1$ still holds, i.e., they are trace-preserving in a general sense. This example may stimulate further studying if general superoperator theory needs modification.

Keywords Operator-sum representation · Kerr medium · Entangled state representation · Dissipative cavity

1 Introduction

For an open quantum system interacting with the environment, one uses superoperators to describe the evolution of density operator from an initial ρ_0 (pure state or mixed state) into a final state ρ . A superoperator plays a role of linear map from $\rho_0 \rightarrow \rho$, which has an operator-sum representation [1–3]

$$\rho = \sum_n M_n \rho_0 M_n^\dagger, \quad (1)$$

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where the operators M_n and M_n^\dagger are usually Hermite conjugate to each other, and obey the normalization condition [4],

$$\sum_n M_n^\dagger M_n = 1. \tag{2}$$

M_n is named Kraus operator [5–7], M_n and M_n^\dagger are Hermite conjugate to each other. Such an operator-sum representation is the core of POVM (positive operator-valued measure).

In our very recent papers [8–11], based on the thermo entangled state representation, we have derived some density operators that are in infinite dimensional operator-sum forms, for example, those for describing amplitude-damping channel and laser process, the corresponding Kraus operators are also obtained and their normalization proved, which implies trace-preserving.

Then an important and interesting question challenges us: is there an infinite operator-sum representation of density operator in the infinite sum form $\rho(t) = \sum_{n=0}^\infty M_n \rho_0 M_n^\dagger$, where the normalization $\sum_{n=0}^\infty M_n^\dagger M_n = 1$ still holds, but M_n and M_n^\dagger are not Hermite conjugate to each other? The answer is affirmative. In this paper, by using the thermo entangled state representation we solve the master equation for a dissipative cavity with Kerr medium to obtain its density operator, and find its infinite operator-sum representation really possesses such a “strange structure”, this may bring attention of theoreticians who might wonder if the general theory of superoperators should be modified.

2 Brief Review of the Thermo Entangled State Representation

In this section, we briefly review the thermo entangled state representation (TESR). Enlightened by the Thermo Dynamic Theory of Takahashi-Umezawa [12–14], we have constructed the TESR in doubled Fock space [15–17],

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right)|0, \tilde{0}\rangle, \tag{3}$$

in which \tilde{a}^\dagger is a fictitious mode accompanying the real photon creation operator a^\dagger , $|0, \tilde{0}\rangle = |0\rangle|\tilde{0}\rangle$, $|\tilde{0}\rangle$ is annihilated by \tilde{a} , $[\tilde{a}, \tilde{a}^\dagger] = 1$. Operating a and \tilde{a} on $|\eta\rangle$ in (3) we obtain the eigen-equations of $|\eta\rangle$,

$$\begin{aligned} (a - \tilde{a}^\dagger)|\eta\rangle &= \eta|\eta\rangle, & (a^\dagger - \tilde{a})|\eta\rangle &= \eta^*|\eta\rangle, \\ \langle\eta|(a^\dagger - \tilde{a}) &= \eta^*\langle\eta|, & \langle\eta|(a - \tilde{a}^\dagger) &= \eta\langle\eta|. \end{aligned} \tag{4}$$

Note that $[(a - \tilde{a}^\dagger), (a^\dagger - \tilde{a})] = 0$, thus $|\eta\rangle$ is the common eigenvector of $(a - \tilde{a}^\dagger)$ and $(\tilde{a} - a^\dagger)$.

Using the normally ordered form of vacuum projector $|0, \tilde{0}\rangle\langle 0, \tilde{0}| = : \exp(-a^\dagger a - \tilde{a}^\dagger \tilde{a}) :$, and the technique of integration within an ordered product (IWOP) of operators [18, 19], we can easily prove that $|\eta\rangle$ is complete and orthonormal,

$$\int \frac{d^2\eta}{\pi} |\eta\rangle\langle\eta| = 1, \quad \langle\eta'|\eta\rangle = \pi\delta(\eta' - \eta)\delta(\eta'^* - \eta^*). \tag{5}$$

The $|\eta = 0\rangle$ state

$$|\eta = 0\rangle = e^{a^\dagger \tilde{a}^\dagger} |0, \tilde{0}\rangle = \sum_{n=0}^\infty |n, \tilde{n}\rangle, \tag{6}$$

possesses the properties [20]

$$\begin{aligned} a|\eta = 0\rangle &= \tilde{a}^\dagger|\eta = 0\rangle, \\ a^\dagger|\eta = 0\rangle &= \tilde{a}|\eta = 0\rangle, \\ (a^\dagger a)^n|\eta = 0\rangle &= (\tilde{a}^\dagger \tilde{a})^n|\eta = 0\rangle. \end{aligned} \quad (7)$$

Note that density operators are functions of (a^\dagger, a) , i.e., defined in the original Fock space, so they are commutative with operators of $(\tilde{a}^\dagger, \tilde{a})$ in the tilde space.

3 Infinite Operator-Sum Representation of Density Operator for a Dissipative Cavity with Kerr Medium

In the Markov approximation and interaction picture the master equation for a dissipative cavity with Kerr medium has the form [21–24]

$$\frac{d\rho}{dt} = -i\chi[(a^\dagger a)^2, \rho] + \gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a), \quad (8)$$

where γ is a decaying parameter of the dissipative cavity, χ is a coupling factor depending on the Kerr medium. Next we will solve the master equation by virtue of the entangled state representation and present the infinite sum representation of density operator.

Operating the both sides of (8) on the state $|\eta = 0\rangle$, letting

$$|\rho\rangle = \rho|\eta = 0\rangle, \quad (9)$$

and using (7) we have the following state vector equation,

$$\frac{d}{dt}|\rho\rangle = \{-i\chi[(a^\dagger a)^2 - (\tilde{a}^\dagger \tilde{a})^2] + \gamma(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a})\}|\rho\rangle. \quad (10)$$

The formal solution of $|\rho\rangle$ is

$$|\rho\rangle = \exp\{-i\chi t[(a^\dagger a)^2 - (\tilde{a}^\dagger \tilde{a})^2] + \gamma t(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a})\}|\rho_0\rangle, \quad (11)$$

where $|\rho_0\rangle = \rho_0|\eta = 0\rangle$, ρ_0 is the initial density operator. By introducing the following operators,

$$K_0 = a^\dagger a - \tilde{a}^\dagger \tilde{a}, \quad K_z = \frac{a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1}{2}, \quad K_- = a\tilde{a}, \quad (12)$$

and noticing $[K_0, K_z] = [K_0, K_-] = 0$, we can rewrite (11) as

$$\begin{aligned} |\rho\rangle &= \exp\{-i\chi t[K_0(2K_z - 1)] + \gamma t(2K_- - 2K_z + 1)\}|\rho_0\rangle \\ &= \exp[i\chi t K_0 + \gamma t] \exp\left\{-2t\left(\gamma + i\chi K_0\right)\left[K_z + \frac{-\gamma}{\gamma + i\chi K_0} K_-\right]\right\}|\rho_0\rangle. \end{aligned} \quad (13)$$

With the aid of the operator identity [25]

$$e^{\lambda(A+\sigma B)} = e^{\lambda A} \exp[\sigma B(1 - e^{-\lambda\tau})/\tau] = \exp[\sigma B(e^{\lambda\tau} - 1)/\tau] e^{\lambda A}, \quad (14)$$

which is valid for $[A, B] = \tau B$, and noticing $[K_z, K_-] = -K_-$, we can reform (13) as

$$|\rho\rangle = \exp[i\chi t K_0 + \gamma t] \exp[\Gamma_z K_z] \exp[\Gamma_- K_-] |\rho_0\rangle, \tag{15}$$

where

$$\Gamma_z = -2t(\gamma + i\chi K_0), \quad \Gamma_- = \frac{\gamma(1 - e^{-2t(\gamma + i\chi K_0)})}{\gamma + i\chi K_0}. \tag{16}$$

In order to deprive of the state $|\eta = 0\rangle$ from (15), using the completeness relation of Fock state in the enlarged space $\sum_{m, \tilde{n}=0}^\infty |m, \tilde{n}\rangle \langle m, \tilde{n}| = 1$ and noticing $a^{\dagger l} |n\rangle = \sqrt{\frac{(n+l)!}{n!}} |n+l\rangle$, we have

$$\begin{aligned} |\rho\rangle &= \exp[i\chi t K_0 + \gamma t] \exp[\Gamma_z K_z] \sum_{l=0}^\infty \frac{\Gamma_-^l}{l!} a^l \rho_0 a^{\dagger l} |\eta = 0\rangle \\ &= \exp[i\chi t K_0 + \gamma t] \sum_{l=0}^\infty \frac{\Gamma_-^l}{l!} \exp[\Gamma_z K_z] \sum_{m, n=0}^\infty |m, \tilde{n}\rangle \langle m, \tilde{n}| a^l \rho_0 a^{\dagger l} |\eta = 0\rangle \\ &= \sum_{m, n, l=0}^\infty \frac{\Lambda_{m, n}^l}{l!} \exp[-i\chi(m^2 - n^2) - t\gamma(m + n)] |m, \tilde{n}\rangle \langle m, \tilde{n}| a^l \rho_0 a^{\dagger l} |\eta = 0\rangle, \end{aligned} \tag{17}$$

where we have set

$$\Lambda_{m, n} \equiv \frac{\gamma(1 - e^{-2t(\gamma + i\chi(m-n))})}{\gamma + i\chi(m - n)}. \tag{18}$$

Further, using

$$\langle n | \eta = 0\rangle = |\tilde{n}\rangle, \quad |m, \tilde{n}\rangle = |m\rangle \langle n | \eta = 0\rangle, \tag{19}$$

which leads to

$$\begin{aligned} \langle m, \tilde{n} | a^l \rho_0 a^{\dagger l} | \eta = 0\rangle &= \langle m | \langle \tilde{n} | a^l \rho_0 a^{\dagger l} | \eta = 0\rangle \\ &= \langle m | a^l \rho_0 a^{\dagger l} (\langle \tilde{n} | \eta = 0\rangle) \\ &= \langle m | a^l \rho_0 a^{\dagger l} | n\rangle, \end{aligned} \tag{20}$$

then (17) becomes

$$\begin{aligned} |\rho\rangle &= \sum_{m, n, l=0}^\infty \frac{\Lambda_{m, n}^l}{l!} \exp[-i\chi t(m^2 - n^2) - \gamma t(m + n)] |m, \tilde{n}\rangle \langle m | a^l \rho_0 a^{\dagger l} | n\rangle \\ &= \sum_{m, n, l=0}^\infty \sqrt{\frac{(n+l)!(m+l)!}{n!m!}} \frac{\Lambda_{m, n}^l}{l!} e^{-i\chi t(m^2 - n^2) - \gamma t(m+n)} |m, \tilde{n}\rangle \rho_{0, m+l, n+l}, \end{aligned} \tag{21}$$

where $\rho_{0, m+l, n+l} \equiv \langle m+l | \rho_0 | n+l\rangle$. Using (19) again, we see

$$|\rho\rangle = \sum_{m, n, l=0}^\infty \sqrt{\frac{(n+l)!(m+l)!}{n!m!}} \frac{(\Lambda_{m, n})^l}{l!} e^{-i\chi t(m^2 - n^2) - \gamma t(m+n)} \rho_{0, m+l, n+l} |m\rangle \langle n | \eta = 0\rangle. \tag{22}$$

After depriving $|\eta = 0\rangle$ from the both sides of (22), the solution of master equation (8) appears as infinite operator-sum form

$$\begin{aligned} \rho(t) &= \sum_{m,n,l=0}^{\infty} \sqrt{\frac{(n+l)!(m+l)!}{n!m!}} \frac{\Lambda_{m,n}^l}{l!} e^{-i\chi t(m^2-n^2)-\gamma t(m+n)} |m\rangle\langle m+l| \rho_0 |n+l\rangle\langle n| \\ &= \sum_{m,n,l=0}^{\infty} \frac{\Lambda_{m,n}^l}{l!} e^{-i\chi t(m^2-n^2)-\gamma t(m+n)} |m\rangle\langle m| a^l \rho_0 a^{\dagger l} |n\rangle\langle n|. \end{aligned} \tag{23}$$

Note that the factor $(m - n)$ appears in the denominator of $\Lambda_{m,n}$ (see (18)), (this is originated from the nonlinear term of $(a^\dagger a)^2$), so moving of all n -dependent terms to the right of $a^l \rho_0 a^{\dagger l}$ is impossible, nevertheless, we can formally express (23) as

$$\rho(t) = \sum_{m,n,l=0}^{\infty} M_{m,n,l} \rho_0 \mathcal{M}_{m,n,l}^\dagger, \tag{24}$$

where the two operators $M_{m,n,l}$ and $\mathcal{M}_{m,n,l}^\dagger$ are respectively defined as

$$\begin{aligned} M_{m,n,l} &\equiv \sqrt{\frac{\Lambda_{m,n}^l}{l!}} e^{-i\chi t m^2 - \gamma t m} |m\rangle\langle m| a^l, \\ \mathcal{M}_{m,n,l}^\dagger &\equiv \left\{ \sqrt{\frac{\Lambda_{n,m}^l}{l!}} e^{-i\chi t n^2 - \gamma t n} |n\rangle\langle n| a^l \right\}^\dagger, \end{aligned} \tag{25}$$

to one’s regret, $M_{m,n,l}$ is not Hermite conjugate to $\mathcal{M}_{m,n,l}^\dagger$. This example may surprise us to wonder if the general superoperator form in (1) needs modification.

4 Further Analysis for $\rho(t)$

Using the operator identity $e^{\lambda a^\dagger a} = : \exp[(e^\lambda - 1)a^\dagger a] :$ and the IWOP technique, we can prove that

$$\begin{aligned} &\sum_{m,n,l=0}^{\infty} \mathcal{M}_{m,n,l}^\dagger M_{m,n,l} \\ &= \sum_{n,l=0}^{\infty} \frac{(n+l)!}{n!} \frac{(1 - e^{-2t\gamma})^l}{l!} e^{-2n\gamma t} |n+l\rangle\langle n+l| \\ &= \sum_{n,l=0}^{\infty} \frac{(1 - e^{-2t\gamma})^l}{l!} a^{\dagger l} e^{-2\gamma t a^\dagger a} |n\rangle\langle n| a^l \\ &= \sum_{l=0}^{\infty} \frac{(1 - e^{-2t\gamma})^l}{l!} : \exp[(e^{-2\gamma t} - 1)a^\dagger a] (a^\dagger a)^l : = 1, \end{aligned} \tag{26}$$

from which one can see that the normalization still holds, i.e., they are trace-preserving in a general sense, so $M_{m,n,l}$ and $\mathcal{M}_{m,n,l}^\dagger$ may be named the generalized Kraus operators.

5 Reduction of $\rho(t)$ in Some Special Cases

In general, (23) indicates that the Kerr medium causes phase diffusion while the field in cavity is in amplitude-damping. In (23), when the decoherent time $t \rightarrow \infty$, the main contribution comes from the $m = n = 0$ term, in this case $\Lambda \rightarrow 1$, then

$$\rho(t \rightarrow \infty) \rightarrow \sum_{l=0}^{\infty} \langle l | \rho_0 | l \rangle |0\rangle \langle 0| = |0\rangle \langle 0|, \tag{27}$$

since $\text{Tr } \rho_0 = 1$, which shows that the quantum system reduces to the vacuum state after a long decoherence interaction, as expected.

In particular, when $\chi = 0$, (23) becomes

$$\begin{aligned} \rho(t) &= \sum_{m,n,l=0}^{\infty} \sqrt{\frac{(n+l)!(m+l)!}{n!m!}} \frac{(1 - e^{-2\gamma t})^l}{l!} e^{-\gamma t(m+n)} \rho_{0,m+l,n+l} |m\rangle \langle n| \\ &= \sum_{m,n,l=0}^{\infty} \frac{(1 - e^{-2\gamma t})^l}{l!} e^{-\gamma t a^\dagger a} |m\rangle \langle m| a^l \rho_0 a^{\dagger l} |n\rangle \langle n| e^{-\gamma t a^\dagger a} \\ &= \sum_{l=0}^{\infty} \frac{(1 - e^{-2\gamma t})^l}{l!} e^{-\gamma t a^\dagger a} a^l \rho_0 a^{\dagger l} e^{-\gamma t a^\dagger a}, \end{aligned} \tag{28}$$

which corresponds to the amplitude decaying mode, coinciding with the result in Refs. [8, 9]. While for $\gamma = 0$, (23) reduces to

$$\rho(t) = \sum_{m,n=0}^{\infty} e^{-i\chi t(m^2-n^2)} |m\rangle \langle m| \rho_0 |n\rangle \langle n| = e^{-i\chi(a^\dagger a)^2 t} \rho_0 e^{i\chi(a^\dagger a)^2 t}, \tag{29}$$

which implies a process of phase diffusion, for $\rho_0 = |n\rangle \langle n|$, then

$$\rho(t) = e^{-i\chi(a^\dagger a)^2 t} |n\rangle \langle n| e^{i\chi(a^\dagger a)^2 t} = |n\rangle \langle n|, \tag{30}$$

which shows neither decay nor phase diffusion happens in the Kerr medium.

In summary, we have demonstrated through the above example that the nonlinear Hamiltonian in master equation may demand us to modify (1), the general expression of superoperator. For more discussions about decoherence in Kerr medium, we refer to Refs. [26–29].

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